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Higher-order interdependent percolation on hypergraphs

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ABSTRACT

A fundamental concern on the robustness of hypergraphs lies in comprehending how the failure of individual nodes affects the hyperedges they are associated with. To address the issue, we propose a simple but novel percolation model that takes into account the dependency of hyperedges on their internal nodes, where the failure of a single node can lead to the dissolution of its associated hyperedge with a probability β. Based on a newly proposed analytical method of percolation theory on hypergraphs, our research reveals that the impact of mean cardinality on the system robustness varies with β. For a large value of β, a larger mean cardinality increases the fragility of hypergraphs, while for a small β, a larger mean cardinality enhances the robustness of hypergraphs. Additionally, our research uncovers divergent effects of hyperdegree distribution on system robustness between monolayer and double-layer hypergraphs. Specifically, monolayer hypergraphs with scale-free hyperdegree distribution exhibit higher robustness, while Poisson hyperdegree distributions lead to stronger robustness in double-layer hypergraphs. These findings provide valuable insights into the robustness of hypergraphs and its dependency on hyperdegree distributions and mean cardinality, contributing to a more comprehensive understanding of the complexities of robustness in complex systems. Furthermore, the development of the percolation model enriches our understanding of node-hyperedge interactions within complex systems.

1. Introduction

Traditional percolation models, such as bond percolation [1,2] or site percolation [3,4] on graphs or networks, have played a crucial role in studying the emergence of collective behaviors in various systems [5,6]. These models have provided valuable insights into the formation of giant connected components and phase transitions in networked systems [7,8]. However, real-world networks often exhibit complex relationships and dependencies that go beyond pairwise connections [9–15]. This limitation restricts the ability of traditional networks to capture the rich higher-order interactions and dynamics of real-world systems.

To address this limitation, hypergraphs offer a more comprehensive representation of complex systems by capturing higher-order interactions and dependencies [10,16–18]. This is due to the fact that hyperedges of a hypergraph can connect more than two nodes, allowing for a more flexible and expressive modeling framework. For instance, in social networks, hypergraphs can capture group interactions [19,20], collaborations [21,22], and communities [23–25] more effectively. In biological networks, hypergraphs can represent protein-protein interactions involving multiple proteins simultaneously [26–28]. In transportation networks, hypergraphs can model traffic flows and interdependencies among different modes of transportation [29–33]. Similarly, multilayer networks also provide a valuable representation of the multiple nature of interactions in complex systems [34,35], which also offers an important framework for capturing the interactions beyond pairwise. Multilayer networks encompass various real-world examples, such as cyber-physical systems [36–38], social systems [39–41], and multilayer traffic networks [42–44].

The robustness of complex systems with higher-order interactions has attracted great attention. One of the most direct approaches to studying the robustness of hypergraphs is through percolation theory, wherein researchers investigate the emergence threshold of the giant component formed by the remaining nodes after the removal of a fraction of nodes [45,46]. These studies have revealed that factors such as the attack strategies [46,47] and degree correlations [48] significantly influence the percolation threshold and robustness of the hypergraph. Moreover, there are some works aimed to understand the dynamics of cascading failures in complex systems under higher-order interactions by modeling coupling mechanisms among nodes in hypergraphs [49,50], coupling between network layers [10] and core percolation on hypergraphs [51].

In addition, over the past decade, the robustness of multilayer networks has been extensively examined through the percolation theory.

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Such investigations have disclosed that node failures within multilayer networks can propagate not only within the individual layers but also across different layers, thereby rendering multilayer networks highly susceptible to discontinuous phase transitions [6,52–54]. Furthermore, it has been discovered that specific topological features and inter-layer coupling characteristics significantly influence the network's robustness, enabling the transition from discontinuous to continuous phase transitions and consequently enhancing its resilience [55–57].

In both monolayer and multilayer networks, higher-order interactions are prevalent and exert a significant influence on system robustness. However, much of the current research implicitly assumes that hyperedges lose their functionality only when they fail to connect two or more nodes. This assumption does not align with real-world observations. In practical scenarios, the departure of a few critical members can trigger the dissolution of entire groups, such as in social communities or collaboration groups. Similarly, in specific infrastructures composed of multiple components, the failure of one or a few components can lead to the malfunction of the entire module. This phenomenon persists when more systems are coupled together to form multilayer networks, as seen in cyber-physical systems, where many power nodes are parallel for electricity generation, forming a collaborative group. The failure of a few nodes can lead to the dissolution of the entire collaborative group, causing partial node failures in the information network. When considering the dependency mechanism of hyperedges on their internal nodes, in the case of monolayer networks, the failure of one or a few nodes within a hyperedge can result in the disintegration of that hyperedge. This, in turn, can cause more nodes in the network to fail, making the system more vulnerable compared to scenarios where this mechanism is not considered. In the context of multilayer networks, node failures not only propagate within a network layer through this mechanism but also, due to the interdependence of nodes across different network layers, can trigger cascading failures that spread across various layers. Consequently, without accounting for the dependency mechanism of hyperedges on their internal nodes, the robustness of the system may be overestimated. However, the dependency mechanism of hyperedges on their internal nodes has received limited attention in previous studies on the robustness of both monolayer and multilayer networks. Therefore, exploring the impact of this mechanism on the robustness or cascading dynamics of complex systems is crucial for both monolayer and multilayer networks. This exploration will enable us to more accurately assess the robustness of real-world complex systems.

In order to investigate the impact of the dependency mechanism of hyperedges on their internal nodes regarding system robustness, we propose a simple percolation model by introducing a tunable model parameter, denoted as β, to control the impact of node failures on the hyperedges. Specifically, when a node fails, the hyperedge it belongs to fails with a probability of β and is retained with a probability of $1-\beta$. This adjustable parameter allows us to investigate the varying degrees of influence that individual node failures have on the integrity and persistence of hyperedges in the complex system. Through a comprehensive analysis of the proposed model, we aim to unveil key insights into the behavior of complex systems with higher-order interdependencies. Our findings reveal important differences between monolayer hypergraphs and double-layer hypergraphs in terms of fragility and phase transitions. Specifically, double-layer hypergraphs consistently exhibit higher fragility compared to monolayer hypergraphs, suggesting that the presence of interdependencies across layers amplifies the vulnerability of complex systems. Interestingly, the impact of hyperdegree distribution on system robustness differs between monolayer hypergraphs and double-layer hypergraphs. In monolayer hypergraphs, a power-law hyperdegree distribution increases robustness, whereas, in double-layer hypergraphs, it enhances fragility. This discrepancy highlights the intricate relationship between hyperdegree distribution and system resilience in different hypergraph types. Additionally, we identify a contrasting effect of mean hyperdegree and mean cardinality on system robustness. A higher mean hyperdegree enhances system resilience, while this effect does not hold for mean cardinalities. Specifically, when β is large, a larger mean cardinality leads to increased fragility in the network. Conversely, when β is small, a larger mean cardinality enhances network robustness. These findings underscore the importance of considering the specific characteristics of hypergraphs when assessing and designing robust systems.

The paper is structured as follows. In Section 2, we present the percolation model of hypergraphs with the dependence of hyperedge on its internal nodes for both Poisson and power-law hyperdegree distributions. In Section 3, we discuss the cascading failure process in the double-layer hypergraph with the dependence of hyperedge on its internal nodes for both Poisson and power-law hyperdegree distributions. In Section 4, we provide the concluding remarks.

2. Percolation on random hypergraphs

2.1. Random hypergraph model

Consider a random hypergraph $H(V,E)$ composed of a set of N nodes, denoted as V, and a set of M hyperedges, denoted as E. Each hyperedge has a distinct cardinality m, i.e., the number of nodes contained in the hyperedge, that follows a distribution $Q(m)$. The hyperdegree k of a node is defined as the number of hyperedges incident to that node, and the distribution of hyperdegrees is represented by $P(k)$. In this paper, we make the simple assumption that the cardinality distribution of hyperedges follows a Poisson distribution with a mean value of $\langle m\rangle$. As for the node hyperdegree distribution, we will separately consider two scenarios: Poisson distribution and power-law distribution.

At the outset, each node in the hypergraph is assigned a failure probability denoted by q, calculated as $q = 1 - p$, where p signifies the node retention probability and lies between the range of 0 to 1. Additionally, we introduce $\beta = 1 - \alpha$ as the probability of hyperedge failure in case a node within that hyperedge is removed. Here, α stands for the probability of hyperedge preservation if one of its internal nodes fails and varies from 0 to 1. In this context, β controls the dependency strength of a hyperedge on its internal nodes. Exactly, as β approaches 0, the dependency of a hyperedge on its internal nodes is weakest. In this scenario, unless all nodes within the hyperedge fail, the failures of a subset of nodes will not cause the hyperedge to collapse. On the other hand, as β tends towards 1, the dependency of a hyperedge on its nodes becomes strongest. The failure of any single node will lead to the dissolution of the entire hyperedge. Under this mechanism, when a fraction of nodes is removed, some hyperedges will also disrupt. The failure process of the random hypergraph can be illustrated in Fig. 1. Our main focus is to investigate the relative size of the giant component in the system, denoted as $S = G/N$, where G represents the size of the giant component. At the same time, we also pay attention to the critical point q_c at which the giant component vanishes in the system. This critical point serves as a metric for the robustness of a system. Specifically, when q_c is large, it implies that the giant component can be destroyed by removing a sufficient fraction of nodes, indicating stronger system robustness. Conversely, when q_c is small, it implies that removing a small fraction of nodes can devastate the giant component, indicating poorer system robustness.

We address the percolation problem on hypergraphs with dependencies of hyperedges on its internal nodes by using the generating function method. Firstly, we introduce two generating functions: $G_0(x) = \sum_{k=0} P(k)x^k$ generates the hyperdegree distribution of nodes and $G_1(x) = \sum_{k=1} kP(k)/\langle k\rangle x^{k-1}$ generates excess hyperdegree distribution of a random node in a hyperedge.

To determine the order parameter S, we introduce an auxiliary parameter R, representing the probability that a randomly chosen hyperedge, reached by a random node, belongs to the giant component. When selecting a random hyperedge through this random node, a

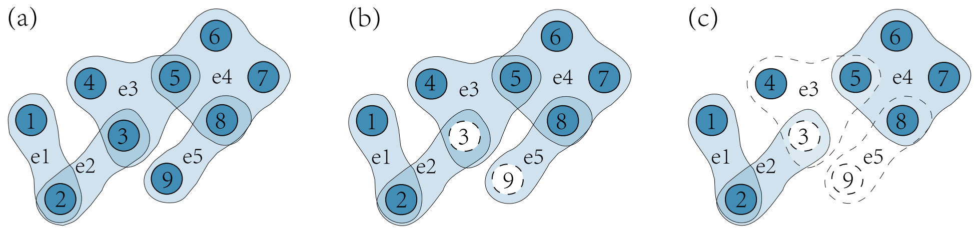

Fig. 1. The illustration demonstrates the failure process on a hypergraph comprising 5 hyperedges, namely $e1, e2, e3, e4, e5$, and 9 vertices, denoted by indices 1 to 9. (a) The original hypergraph is depicted without any attacks or failures; (b) Initially, we start by removing nodes 3 and 9; (c) It can be observed that the failure of node 3 and 9 has led to the malfunction of hyperedges $e3$ and $e5$, respectively.

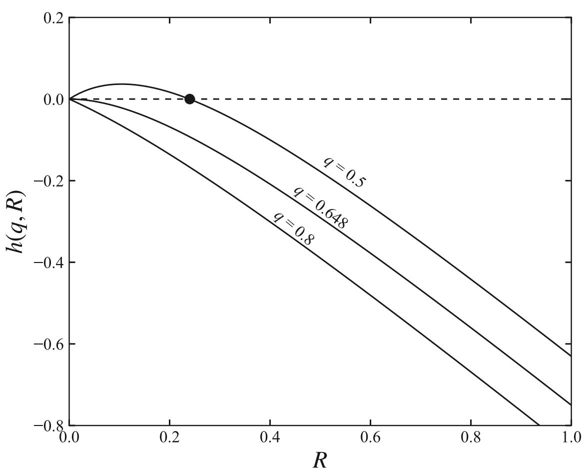

Fig. 2. The graphical representation of Eq. (4) is shown for different fractions q of nodes initially removed. The graph illustrates the results of a monolayer hypergraph with $\beta = 0.4$. Both the hyperdegree distribution and cardinality distribution of the hypergraph follow Poisson distributions with $\langle m\rangle = 4$ and $\langle k\rangle = 2$, respectively.

hyperedge with a larger cardinality is more likely to be selected. Therefore, the probability of randomly selecting a hyperedge with cardinality m is proportional to the product of its probability $Q(m)$ and its cardinality m, and the probability therefore can be calculated as $Q(m)m/\langle m\rangle$. Further, excluding the randomly chosen node, the probability that the reached hyperedge still survives when n nodes fail among the $m-1$ nodes, can be expressed as $(1-\beta)^n\binom{m-1}{n}(1-q)^{m-1-n}q^n$. Since the excess hyperdegree distribution of a random node is generated by $G_1(x)$, the distribution of the number of hyperedges that the randomly chosen hyperedge can connect to through these remaining $m-1-n$ nodes can be given by the generating function $G_1^{m-1-n}(x)$. Thus, the probability of this hyperedge being connected to the giant component is $1-G_1^{m-1-n}(1-R)$. By the above considerations, the auxiliary parameter R can be obtained as

$$R = \sum_m \frac{mQ(m)}{\langle m\rangle}\sum_{n=0}^{m-1}(1-\beta)^n\left[1-G_1^{m-1-n}(1-R)\right]\binom{m-1}{n}(1-q)^{m-1-n}q^n. \quad (1)$$

Since the probability that a randomly chosen node cannot reach the giant component is $G_0(1-R)$, then the order parameter S can be expressed as

$$S = (1-q)\left[1-G_0(1-R)\right]. \quad (2)$$

To solve R, we here define the following equation

$$h(q, R) = 0 \quad (3)$$

with the function defined as

$$h(q, R) = \sum_m \frac{mQ(m)}{\langle m\rangle}\sum_{n=0}^{m-1}(1-\beta)^n\left[1-G_1^{m-1-n}(1-R)\right]\binom{m-1}{n} \times (1-q)^{m-1-n}q^n - R. \quad (4)$$

As shown in Fig. 2, when the function curve is tangent with the R-axis, a nontrivial solution of R emerges, and thus the critical value of q_c^{II} can be obtained by

$$\partial_R h(q_c^{II}, 0) = 0. \quad (5)$$

Thus, we can get

$$(1-q_c^{II})G_1'(1)\sum_m \frac{mQ(m)}{\langle m\rangle}(m-1)\left[1-q_c^{II}+(1-\beta)q_c^{II}\right]^{m-2} = 1. \quad (6)$$

Our theoretical framework also provides a more concise and elegant solution to the percolation problem on hypergraphs. If $\beta = 0$, our model reduces to the case of ordinary percolation on hypergraphs. According to Eq. (6), we can get

$$q_c^{II} = 1 - \frac{\langle m\rangle}{\langle m(m-1)\rangle G_1'(1)}. \quad (7)$$

This result is consistent with the findings reported in [10].

Since cardinality m follows Poisson distribution (i.e., $Q(m) = \frac{e^{-\langle m\rangle}\langle m\rangle^m}{m!}$), based on Eq. (6), we can further obtain

$$\langle m\rangle G_1'(1)(1-q_c^{II})e^{-\langle m\rangle\beta q_c^{II}} = 1. \quad (8)$$

By substituting the parameters $\beta = 0.4$, $\langle m\rangle = 4$ and $\langle k\rangle = 2$ into Eq. (8), we can estimate the critical value q_c^{II} to be approximately 0.648. This calculation provides an estimation of the threshold value at which a second-order phase transition occurs in the hypergraph. The confirmation of this critical value can be done by referring to Fig. 2.

2.2. Poisson degree distribution

We first consider the case of a Poisson hyperdegree distribution $P(k) = \frac{e^{-\langle k\rangle}\langle k\rangle^k}{k!}$, with $\langle k\rangle$ representing the mean hyperdegree. In this scenario, as the model parameter β increases, the system becomes more fragile and leads to an decrease in the critical point q_c^{II}, indicating that the system becomes more susceptible to initial failures as the increased dependence of hyperedges on its internal nodes, which is illustrated by the phase diagram in Fig. 3(a–b). To confirm the position of the critical point q_c^{II}, we plot several cross-sections of the phase diagram at different values of β in Fig. 3(c), and analyze the function curves of susceptibility χ versus q for different β in Fig. 3(d), where the susceptibility is calculated as $\chi = \frac{\langle G^2\rangle - \langle G\rangle^2}{\langle G\rangle}$. It can be observed that χ peaks at the point where the giant component disappears. These results indicate that the model parameter β has a monotonous effect on the robustness of the hypergraph, meaning that a larger β leads to poorer system robustness. Additionally, we observe a high level of consistency between our simulation results and theoretical findings, providing further validation of the accuracy of our theory.

2.3. Power-law hyperdegree distribution

We next consider the case where the hyperdegree distribution follows a scale-free power law, represented by $P(k) = Ck^{-\gamma}$ within the range $k_{min} \le k \le k_{max}$. Here, k_{min} and k_{max} denote the lower and upper

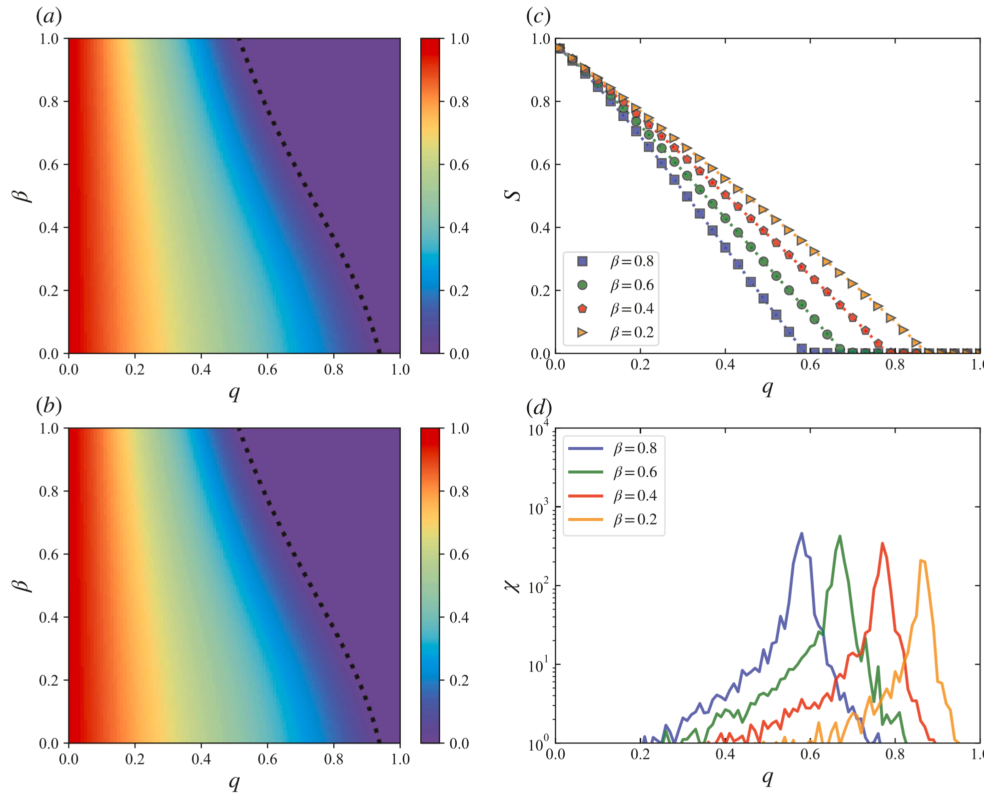

Fig. 3. Percolation transition on random hypergraphs. Panels (a) and (b) illustrate the simulated and theoretical phase diagrams for percolation transitions on random hypergraphs with $\langle m\rangle = 4$ and $\langle k\rangle = 4$ respectively, where the dotted line outlines the theoretical percolation transition point q_c^{II}. In (c), cross-sections of the phase diagram are depicted, illustrating S versus q for different β. Symbols represent simulation data, and dotted lines represent theoretical predictions. Panel (d) displays the corresponding susceptibility χ from simulations. The simulation results are obtained from 50 independent runs, and the hypergraph size is $N = 10^5$.

bounds of the hyperdegree distribution, respectively, and γ represents the power coefficient. Comparing this case to the Poisson distribution, we also observe that the hypergraph also disintegrates as a second-order phase transition with the susceptibility χ reaching its peak at the percolation point, as shown in Fig. 4.

Under the same average hyperdegree, average cardinality, and β, hypergraphs with a power-law hyperdegree distribution demonstrate higher robustness compared to those with a Poisson distribution. This can be observed by comparing Figs. 3 and 4, where scale-free hypergraphs consistently have a larger q_c for the same β. This suggests that a hypergraph with a power-law structure, characterized by hubs or highly connected nodes, is more robust to random failures. The presence of a few hub nodes that enhance the connectivity of the hypergraph. As a result, the hypergraph with power-law hyperdegree distribution maintains its overall connectivity and robustness in the face of random failures.

In addition, we explore the dependence of the percolation point q_c^{II} on the model parameter β for both random hypergraph and power-law hypergraph, considering different mean cardinalities $\langle m\rangle$ and mean hyperdegrees $\langle k\rangle$. As shown in Fig. 5, it is observed that higher values of β consistently lead to lower critical points q_c^{II} for both random hypergraph and power-law hypergraph, indicating decreased robustness of the system. In addition, higher values of the mean hyperdegree $\langle k\rangle$ correspond to a higher level of system robustness (see Fig. 5(a)). However, for the mean cardinality $\langle m\rangle$, a larger mean cardinality leads to increased fragility of the hypergraph when β is large. This is because, when the hyperedges contain more nodes, they are more likely to disintegrate due to the failure of their internal nodes, thus reducing the robustness of the network. However, when β tends towards 0, hyperedges are not prone to disintegrate and a larger mean cardinality $\langle m\rangle$ can improve the connectivity of the network, which in turn improves the robustness of the network (see Fig. 5(b)).

3. Cascading failures on double-layer hypergraphs

3.1. Random double-layer hypergraph

We consider a double-layer hypergraph, where each of the two layers A and B contains N nodes and M hyperedges. Each node in layer A has a corresponding interdependent node in layer B. When a node in layer A fails, it immediately triggers the failure of its interdependent node in layer B, and vice versa. To initiate the cascading failure process, an initial fraction of nodes, q, is removed from the system. Additionally, we assume that the dissolution probability of each hyperedge is β given the removal of a node within that hyperedge. In our study, we assume that only nodes belonging to the giant component in each layer can survive. Both node removal and hyperedge dissolution may lead to further fragmentation of the giant component, causing more nodes to disconnect from the giant component and subsequently fail. Due to the interdependencies between nodes in different network layers, the failure of nodes in layer A will propagate to layer B, causing further fragmentation in layer B and leading to the failure of more nodes. These failed nodes will then spread back to layer A. This iterative process continues, and the system eventually reaches a steady state with the final survival nodes being in the giant component of their respective layers and forming a mutual giant connected component (MGCC). The cascading failure process in the double-layer hypergraph is illustrated in Fig. 6. Under the influence of such cascading failures, the system undergoes a first-order phase transition. We focus on the fraction S_{AB} of nodes in the MGCC and the corresponding threshold q_c^{II} at which the MGCC vanishes.

The generation functions for hyperdegree distribution and excess hyperdegree distribution in both layers A and B can be defined as

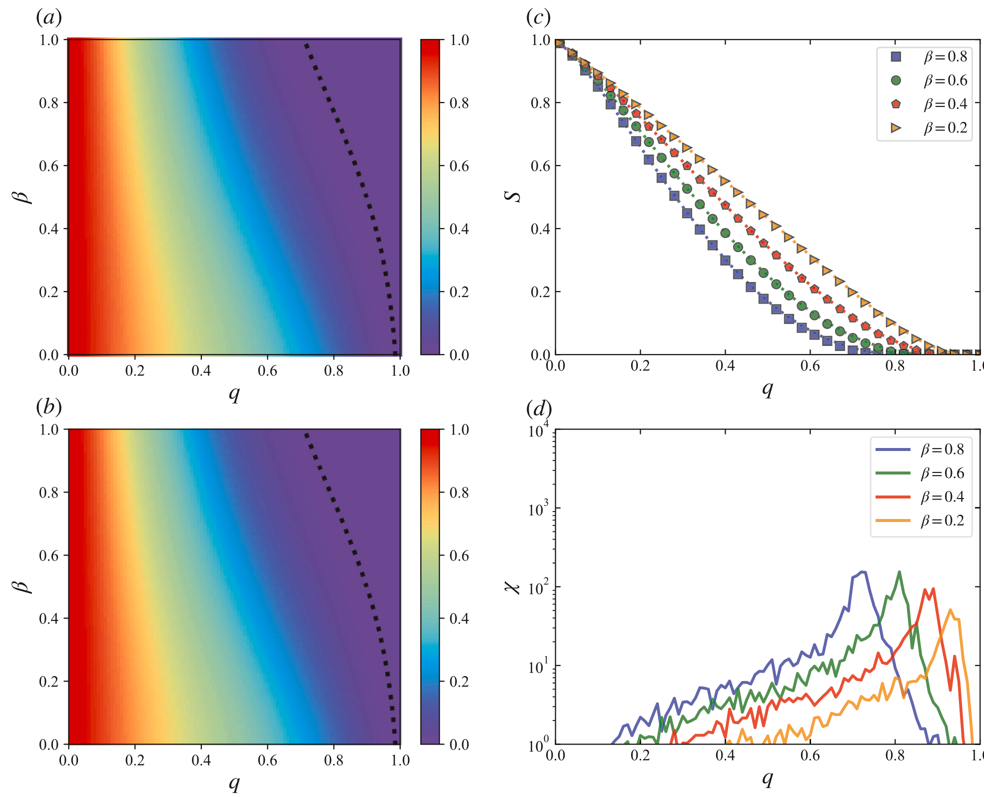

Fig. 4. Percolation transition on scale-free hypergraphs. Panels (a) and (b) illustrate the simulated and theoretical phase diagrams for percolation transitions on scale-free hypergraphs with $\langle m\rangle = 4$ and $\langle k\rangle = 4(\gamma = 2.6, k_{min} = 2, k_{max} = 227)$ respectively, where the dotted line outlines the theoretical percolation transition point q_c^{II}. In (c), cross-sections of the phase diagram are depicted, illustrating S versus q for different β. Symbols represent simulation data, and dotted lines represent theoretical predictions. Panel (d) displays the corresponding susceptibility χ from simulations. The simulation results are obtained from 50 independent runs, and the hypergraph size is $N = 10^5$.

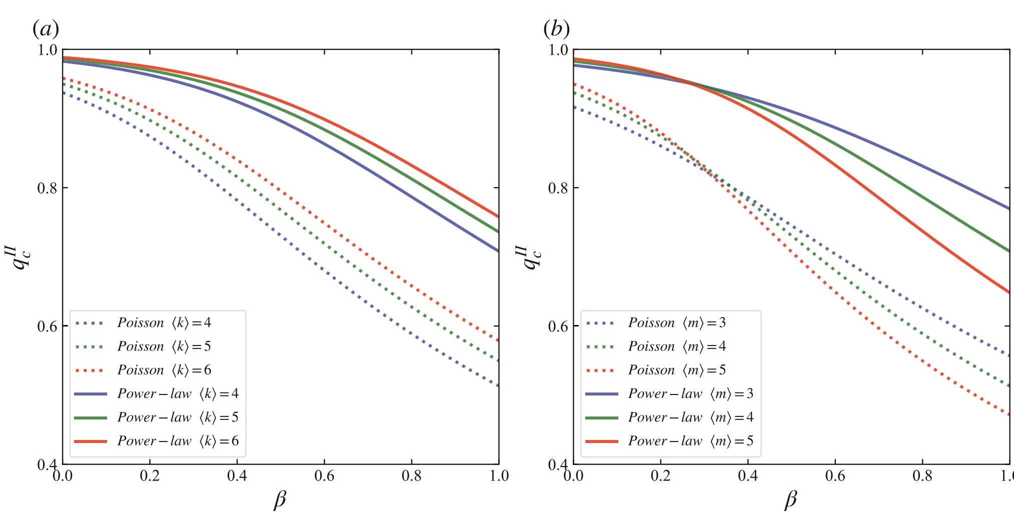

Fig. 5. The relationship between the transition point q_c^{II} and the model parameter β under various average degrees $\langle k\rangle$ and average cardinality $\langle m\rangle$ for random hypergraphs and scale-free hypergraphs. Panel (a) presents the results for $\langle k\rangle = 4, 5, 6$, and a fixed $\langle m\rangle = 4$. Panel (b) shows the results for $\langle m\rangle = 3, 4, 5$, and a fixed $\langle k\rangle = 4$. The dotted lines represent the theoretical results for hypergraphs with a random degree distribution, while the solid lines depict the theoretical results for hypergraphs with a power-law degree distribution. For scale-free hypergraphs, the average hyperdegrees $\langle k\rangle = 4, 5, 6$ correspond to exponents $\gamma = 2.6, 2.3, 2.1$ for degree ranges $(k_{min}, k_{max}) = (2, 227), (2, 128), (2, 109)$, respectively.

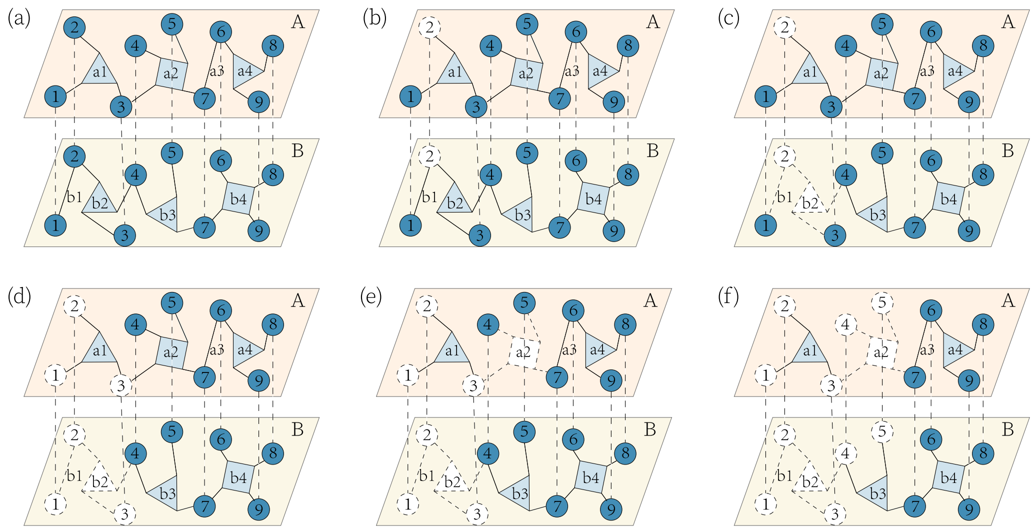

Fig. 6. The illustration of cascading failure process on a double-layer hypergraph. (a) represents the original hypergraph with 9 nodes and 8 hyperedges; (b) Node 2 is removed from layer A as an initial action. As a consequence, the corresponding dependent node 2 in layer B fails; (c) The failure of node 2 leads to the dissolution of hyperedges $b1$ and $b2$ in layer B due to their dependence on node 2; (d) Furthermore, nodes 1 and 3 in layer B are disconnected from the giant component, resulting in the failure of the corresponding interdependent nodes 1 and 3 in layer A; (d) Node 3 triggers the dissolution of hyperedge $a2$ in layer A; (f) Finally, nodes 4 and 5 fail as they become isolated from the giant component in layer A, leading to the removal of their corresponding dependent nodes 4 and 5 in layer B. The cascading failure process ceases with the survival of the MGCC containing nodes 6 to 9.

follows

$$
\begin{cases}
G_0^A(x) = \sum_{k_A=0} P_A(k_A) x^{k_A}, \\
G_0^B(x) = \sum_{k_B=0} P_B(k_B) x^{k_B}, \\
G_1^A(x) = \sum_{k_A=1} \dfrac{P_A(k_A) k_A}{\langle k_A \rangle} x^{k_A - 1}, \\
G_1^B(x) = \sum_{k_B=1} \dfrac{P_B(k_B) k_B}{\langle k_B \rangle} x^{k_B - 1}.
\end{cases}
$$

To solve the order parameter S_{AB}, several auxiliary parameters are defined. R_A and R_B represent the probabilities that a randomly chosen hyperedge of a random node in layers A and B can connect to their giant components respectively. $S_{A\to B}$ ($S_{B\to A}$) represents the probability that the interdependent node in layer $B(A)$ of a random node in layer $A(B)$ can connect to the giant component. Thus $S_{A\to B}$ and $S_{B\to A}$ can be obtained as

$$
\begin{cases}
S_{A\to B} = (1-q)\left[1 - G_0^B(1 - R_B)\right], \\
S_{B\to A} = (1-q)\left[1 - G_0^A(1 - R_A)\right].
\end{cases} \tag{9}
$$

The recursive equations for R_A and R_B can be expressed as

$$
\begin{cases}
R_A = \sum_m \dfrac{m Q_A(m)}{\langle m \rangle} \sum_{n=0}^{m-1} (1-\beta)^n \left(1 - \left[G_1^A(1 - R_A)\right]^{m-1-n}\right) \\
\quad \times \dbinom{m-1}{n} S_{A\to B}^{m-1-n} (1 - S_{A\to B})^n, \\
R_B = \sum_m \dfrac{m Q_B(m)}{\langle m \rangle} \sum_{n=0}^{m-1} (1-\beta)^n \left(1 - \left[G_1^B(1 - R_B)\right]^{m-1-n}\right) \\
\quad \times \dbinom{m-1}{n} S_{B\to A}^{m-1-n} (1 - S_{B\to A})^n.
\end{cases} \tag{10}
$$

Thus the fraction S_{AB} of nodes in MGCC can be given by

$$
S_{AB} = (1-q)\left[1 - G_0^A(1 - R_A)\right]\left[1 - G_0^B(1 - R_B)\right].
$$

In this paper, we only consider the most simple scenario, where the two layers, A and B, have identical hyperdegree distributions (i.e., $P_A(k) = P_B(k)$) and cardinality distributions (i.e., $Q_A(m) = Q_B(m)$). Taking the Poisson hyperdegree distribution and the Poisson cardinality distribution as examples, we have observed that the function curves of Eq. (10) could be tangent at a point q_c^I with the decrease of q for a given β. Based on this observation, we can infer that for double-layer hypergraphs, the breakdown of MGCC occurs in a first-order phase transition, and the critical value q_c^I can be estimated according to the tangent point (confirmed by Fig. 7).

3.2. Poisson hyperdegree distribution

We first consider the case where the hyperdegree follows Poisson distribution for both two layers (i.e., $P_A(k) = P_B(k) = \frac{e^{-\langle k \rangle} \langle k \rangle^k}{k!}$). Compared to monolayer hypergraphs, double-layer hypergraphs exhibit higher fragility, which can be observed in the percolation manner of the systems and the critical points. For double-layer hypergraphs, the percolation phase transition type is a first-order discontinuous phase transition. In contrast, for monolayer hypergraphs, the percolation phase transition type is a second-order continuous phase transition. Additionally, the critical point of the percolation phase transition in monolayer hypergraphs is larger than that in double-layer hypergraphs (as shown in Figs. 3 and 8). These differences highlight the contrasting nature of phase transitions between monolayer and double-layer hypergraphs, emphasizing the importance of interlayer interdependencies in shaping the robustness of complex systems.

3.3. Power-law hyperdegree distribution

We consider the scenario where the hyperdegree distribution follows a power-law distribution for both layers of the double-layer hypergraphs. Specifically, we assume that the hyperdegree distributions for layer A and layer B are given by $P_A(k) = P_B(k) = Ck^{-\gamma}$, where C is a normalization constant and γ is the exponent of the power-law distribution. In this case, we still observe a first-order phase transition in the system, as depicted in Fig. 9. However, an important observation is that the system of double-layer hypergraphs with Poisson hyperdegree distribution is more robust than the system with power-law hyperdegree distribution. This means that the double-layer system

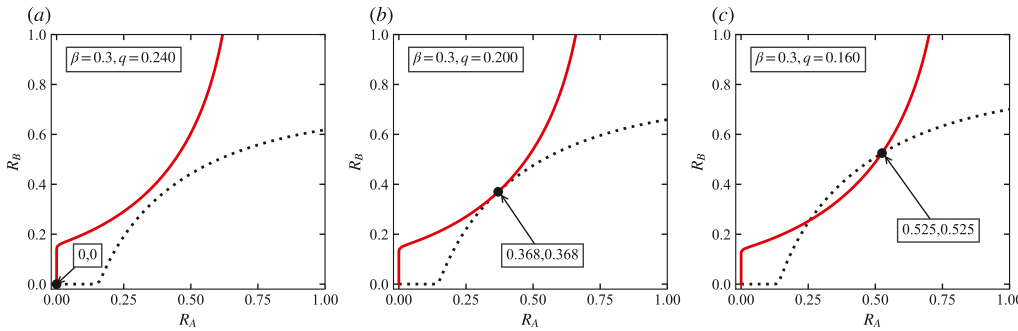

Fig. 7. The solutions of Eq. (10) for a fixed $\beta = 0.3$. Settings: (a) $q = 0.240$; (b) $q = 0.200$; (c) $q = 0.160$. The mean cardinality is $\langle m \rangle = 4$, and the mean hyperdegree is $\langle k \rangle = 2.5$.

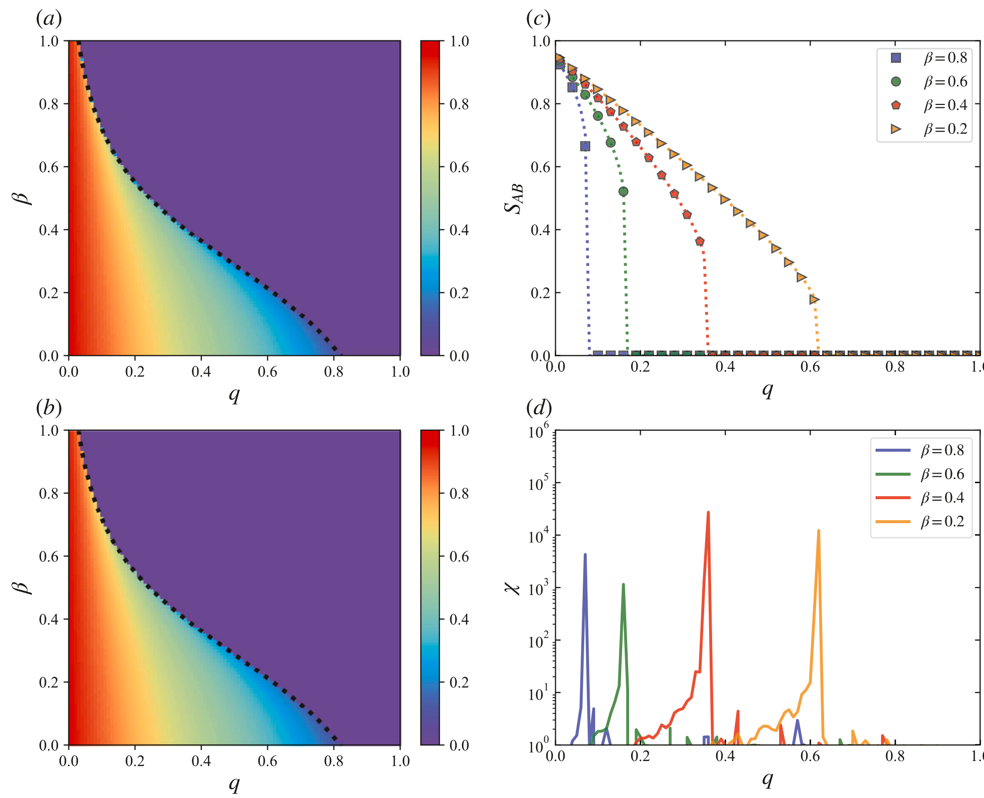

Fig. 8. Percolation transition on double-layer random hypergraphs. Panels (a) and (b) illustrate the simulated and theoretical phase diagrams for percolation transitions on random hypergraphs with $\langle m \rangle = 4$ and $\langle k \rangle = 4$ respectively, where the dotted line outlines the theoretical percolation transition point q_c^I. In (c), cross-sections of the phase diagram are depicted, illustrating S versus q for different β. Symbols represent simulation data, and dotted lines represent theoretical predictions. Panel (d) displays the corresponding susceptibility χ from simulations. The simulation results are obtained from 50 independent runs, and the hypergraph size is $N = 10^5$.

with Poisson hyperdegree distribution is more resilient or less sensitive to random perturbations.

To support this observation, we can refer to Fig. 10, which compares the curves of the percolation transition point q_c^I versus β of double-layer hypergraphs with Poisson and power-law hyperdegree distribution for different parameter settings. By comparing the results, we can observe that the percolation point q_c^I for double-layer scale-free hypergraphs is lower than that for double-layer random hypergraphs for the same mean hyperdegree $\langle k \rangle$, mean cardinality $\langle m \rangle$ and dependency strength β. This indicates that relatively, double-layer scale-free hypergraphs exhibit poorer robustness. Furthermore, larger average degrees are associated with larger q_c^I, implying that higher average hyperdegrees lead to stronger network robustness. Lastly, whether it is for the Poisson degree distribution or the scale-free degree distribution, when β is large, smaller values of $\langle m \rangle$ result in stronger network robustness, while when β is small, larger values of $\langle m \rangle$ lead to stronger robustness.

4. Conclusion

In this paper, we investigated the robustness of complex systems under higher-order interdependencies using hypergraph models. We developed a percolation model that considered the interactions between nodes and hyperedges in hypergraphs and studied the robustness of hypergraphs under the mechanism that node failure would lead to the dissolution of hyperedge in both monolayer and double-layer hypergraphs. Our analysis reveals several important findings that contribute to our understanding of complex systems and their resilience in the face of higher-order interdependencies.

Firstly, we observed that double-layer hypergraphs consistently exhibit greater fragility compared to monolayer hypergraphs, regardless of the hyperdegree distribution (Poisson or power-law). This result suggests that the presence of interdependencies across layers in double-layer hypergraphs amplifies the potential for cascading failures, making

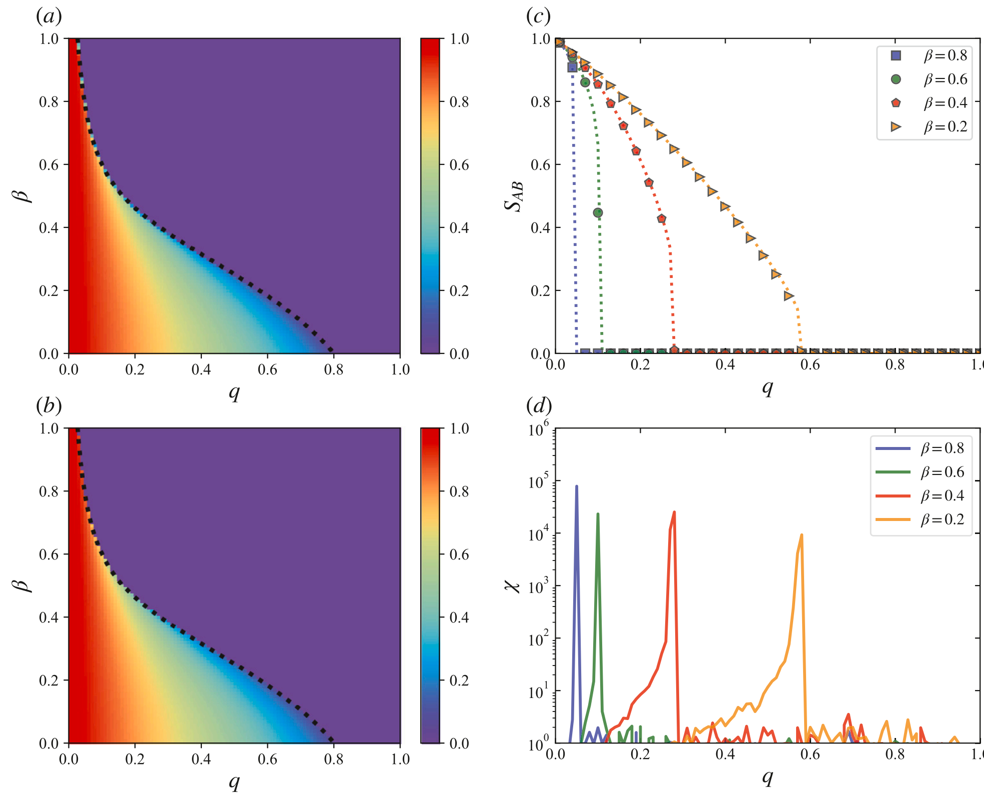

Fig. 9. Percolation transition on double-layer scale-free hypergraphs. Panels (a) and (b) illustrate the simulated and theoretical phase diagrams for percolation transitions on random hypergraphs with $\langle m\rangle = 4$ and $\langle k\rangle = 4(\gamma = 2.6, k_{min} = 2, k_{max} = 227)$ respectively, where the dotted line outlines the theoretical percolation transition point q_c^I. In (c), cross-sections of the phase diagram are depicted, illustrating S versus q for different β. Symbols represent simulation data, and dotted lines represent theoretical predictions. Panel (d) displays the corresponding susceptibility χ from simulations. The simulation results are obtained from 50 independent runs, and the hypergraph size is $N = 10^5$.

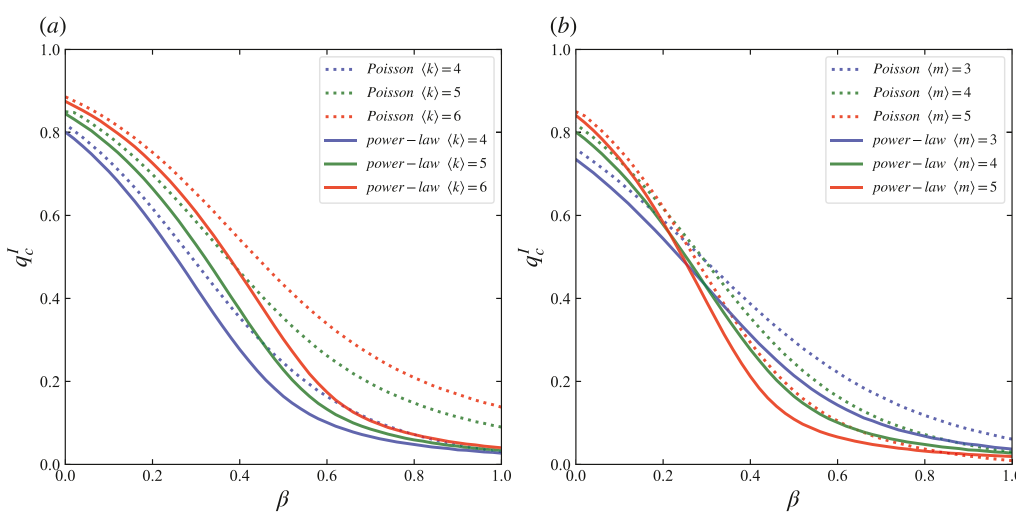

Fig. 10. The relationship between the transition point q_c^I and the model parameter β under various average degrees $\langle k\rangle$ and average cardinality $\langle m\rangle$ for double-layer random hypergraphs and scale-free hypergraphs. Panel (a) presents the results for $\langle k\rangle = 4, 5, 6$, and a fixed $\langle m\rangle = 4$. Panel (b) shows the results for $\langle m\rangle = 3, 4, 5$, and a fixed $\langle k\rangle = 4$. The dotted lines represent the theoretical results for hypergraphs with a random degree distribution, while the solid lines depict the theoretical results for hypergraphs with a power-law degree distribution. For scale-free hypergraphs, the average hyperdegrees $\langle k\rangle = 4, 5, 6$ correspond to exponents $\gamma = 2.6, 2.3, 2.1$ for degree ranges $(k_{min}, k_{max}) = (2, 227), (2, 128), (2, 109)$, respectively.

these systems more vulnerable to random perturbations [52]. This finding emphasizes the critical role of interdependencies in shaping system robustness and calls for attention when designing and managing real-world systems characterized by multilayer interconnections.

Additionally, our study also revealed the divergent effects of hyperdegree distribution on system robustness in monolayer hypergraphs and multilayer hypergraphs. In monolayer hypergraphs, a power-law hyperdegree distribution reduces system fragility. This indicates that highly connected nodes with large hyperdegrees play a crucial role in maintaining the overall connectivity of the hypergraph. Specifically, nodes with large hyperdegrees are rare in the system, which makes them difficult to be removed in a random attack, thus ensuring the connectivity and robustness of the system. This result is consistent with findings in ordinary networks [58]. On the other hand, in double-layer hypergraphs, a power-law hyperdegree distribution diminishes system robustness. Specifically, there is a cascading failure effect where the scale of failures can amplify iteratively in double-layer hypergraphs, and nodes are susceptible to failure due to the failure of their interdependent nodes, including hub nodes with large hyperdegrees. This scenario significantly reduces the network's connectivity. These contrasting effects underscore the intricate relationship between hyperdegree distribution and system behavior in different types of hypergraphs, emphasizing the importance of a thorough understanding of interdependencies when analyzing and designing complex systems.

Furthermore, our analysis reveals the contrasting effect of mean hyperdegree and mean cardinality on system robustness. Whether it is a power-law hyperdegree distribution or a random hyperdegree distribution, a higher mean hyperdegree enhances system robustness, as it increases the number of hyperedges a node participates in and enhances the connectivity of hypergraphs. However, this effect does not hold for mean cardinalities, when β is large, smaller values of mean cardinalities result in stronger network robustness, while when β is small, larger values of mean cardinalities lead to stronger robustness. This finding underscores the importance of considering both hyperdegrees and hyperedge cardinalities when evaluating system resilience.

Our research provides valuable insights into the behavior of complex systems under higher-order interdependencies. Understanding the fragility of monolayer or double-layer hypergraphs and the contrasting effects of hyperdegree distribution and connectivity density on robustness will aid in the design and management of resilient systems. This has broad implications for network science, computational modeling, and resilience analysis, enabling us to assess and design robust systems effectively in real-world applications. By shedding light on the intricate relationship between higher-order interdependencies and system robustness, this study contributes to the advancement of our understanding of their behavior in interconnected environments. Looking ahead, it is important to note that real-world multilayer networks often exhibit varying degrees of correlation in the coupling between layers. Whether highly connected nodes in one layer are coupled with similarly connected nodes in another layer or vice versa, these correlations have a substantial impact on the robustness of the multilayer network. Future research should delve deeper into considering additional topological properties and coupling characteristics of hypergraphs to comprehensively understand their influence on system robustness, further advancing our understanding of the robustness of complex systems.

CRediT authorship contribution statement

Run-Ran Liu: Methodology, Theoretical analysis, Visualization, Writing – original draft. **Changchang Chu:** Simulation, Investigation, Visualization. **Fanyuan Meng:** Writing – review & editing, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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